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Laplace Operator and Random Walk on One-Dimensional Nonhomogeneous Lattice

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A classical result of probability theory states that under suitable space and time renormalization, a random walk converges to Brownian motion. We prove an analogous result in the case of nonhomogeneous random walk on one-dimensional lattice. Under suitable conditions on the nonhomogeneous medium, we prove convergence to Brownian motion and explicitly compute the diffusion coefficient. The proofs are based on the study of the spectrum of random matrices of increasing dimension.

KEY WORDS: Random walk on nonhomogeneous lattice; spectrum of a random matrix.

1. INTRODUCTION

Let $a_0, a_1, a_2, \ldots$ be a sequence of positive numbers. Consider an operator $A^{(N)}: \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ which acts on the vector $u^{(N)} = (u_1^{(N)}, u_2^{(N)}, \ldots, u_{N-1}^{(N)})$ in the following way:

$$(A^{(N)}u^{(N)})_k = (1 - \delta_{k-1})a_{k-1}u_{k-1}^{(N)} - (a_{k-1} + a_k)u_k^{(N)} + (1 - \delta_{N-k-1})a_k u_{k+1}^{(N)}$$

where

$$\delta_k = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

This operator may be treated as the analog of the Laplacian and the equation

$$\dot{u}^{(N)} = A^{(N)}u^{(N)} \tag{1}$$

as a diffusion equation on a nonhomogeneous lattice with the nodes

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located at the points $1/N, 2/N, \ldots, (N-1)/N$ and with absorbing barriers at the points 0 and 1. The sequence $a_0, a_1, a_2, \ldots$ may be treated as sequence of random variables. In this case Eq. (1) describes a diffusion on the lattice with random transport coefficients.

In the present paper we consider the asymptotic behavior when N tends to infinity. Our main result is that, under suitable conditions of macroscopic homogeneity formulated below and an appropriate scaling of time the solutions of Eq. (1) tend to the corresponding solutions of the diffusion equation

$$\dot{u} = au''$$

on the interval (0, 1) with the boundary conditions $u(0) = u(1) = 0$, where

$$a = \lim_{N \to +\infty} N\left(\sum_{k=0}^{N-1} a_k^{-1}\right)^{-1} \tag{2}$$

Using more physical language we can say that a self-averaging of the one-dimensional nonhomogeneous medium takes place and the formula (2) gives the way to compute the macroscopical diffusion coefficient of the limit homogeneous medium.

2. ASSUMPTIONS AND RESULTS

Let us assume that the sequence $a_0, a_1, a_2, \ldots$ satisfies the following condition of macroscopic homogeneity:

E. The following limit exists and is positive:

$$a = \lim_{N \to +\infty} N\left(\sum_{k=0}^{N-1} a_k^{-1}\right)^{-1}$$

Examples of sequences satisfying the above condition are given by the typical realizations of stationary random processes for which $0 < Ea_k^{-1} < +\infty$. In this case the condition E holds almost everywhere. If the random process is also ergodic then a is constant almost everywhere, namely,

$$a = \left(Ea_k^{-1}\right)^{-1}.$$

We also need the following condition:

B. $a_k \geqslant c > 0$ for all k, which is of a more technical character.

Let $\lambda_1^{(N)}, \lambda_2^{(N)}, \ldots, \lambda_{N-1}^{(N)}$ be the eigenvalues of the operator $A^{(N)}$ numbered in nonincreasing order, $\varphi^{(N),j}$ be the eigenvector of the operator $A^{(N)}$ corresponding to the eigenvalue $\lambda_j^{(N)}$ such that $\varphi_1^{(N),j} \geqslant 0$ and

$N^{-1}\sum_{k=1}^{N-1}(\varphi_k^{(N),j})^2 = 1$. To the vector $\varphi^{(N),j} \in \mathbb{R}^{N-1}$ let us associate the step function defined on the interval (0, 1):

$$\varphi^{(N),j}(\xi) = \begin{cases} \varphi_{[N\xi]}^{(N),j}, & N^{-1} \leqslant \xi < 1 \\ 0, & 0 < \xi < N^{-1} \end{cases} \tag{3}$$

We prove the following theorem:

Theorem 1. If condition E holds then

(a) $$\lim_{N\to+\infty} N^2\lambda_j^{(N)} = -a\pi^2 j^2$$

(b) $$\lim_{N\to+\infty} \varphi^{(N),j}(\xi) = \sqrt{2}\sin(\pi j\xi)$$

and the convergence is uniform in $\xi \in (0, 1)$. Moreover for every j there exists a number $N(j)$ such that for every $N > N(j)$ the eigenvalue $\lambda_j^{(N)}$ is nondegenerated.

Let $(A^{(N)})^{-1}$ be defined by the matrix $\|r_{k,m}^{(N)}\|$. To this matrix corresponds the step function

$$r^{(N)}(\xi,\eta) = \begin{cases} N^{-1}r_{[N\xi],[N\eta]}^{(N)}, & N^{-1} \leqslant \xi < 1 \quad \text{and} \quad N^{-1} \leqslant \eta < 1 \\ 0, & 0 < \xi < N^{-1} \quad \text{or} \quad 0 < \eta < N^{-1} \end{cases} \tag{4}$$

defined on the square $(0, 1) \times (0, 1)$.

Theorem 2. If condition E holds then

$$\lim_{N\to+\infty} r^{(N)}(\xi,\eta) = a^{-1}(\xi\eta - \min(\xi,\eta))$$

and the convergence is uniform on the square $(0, 1) \times (0, 1)$.

Let T_t, $t \geqslant 0$ be the semigroup of operators generated by the diffusion equation

$$\begin{aligned} \dot{u} &= au'' \\ u(0) &= u(1) = 0 \end{aligned} \tag{5}$$

on the interval (0, 1) with zero boundary conditions (see Ref. 1). It acts on the space M of finite measures on the interval (0, 1). If μ is some finite measure on the interval (0, 1) then $\mu_t = T_t\mu$.

As a Cauchy initial condition for Eq. (1) we take the vector $\mu^{(N)} = (\mu_1^{(N)}, \mu_2^{(N)}, \ldots, \mu_{N-1}^{(N)}) \in \mathbb{R}^{N-1}$, where

$$\mu_k^{(N)} = N\mu([k/N, (k+1)/N)), \qquad k = 1, 2, \ldots, N-1 \tag{6}$$

Let $\underset{\cdot}{u}^{(N)}(t)$ be a solution of Eq. (1) with such initial condition. As above we define the step function on the interval (0, 1)

$$u^{(N)}(\xi,t)=\begin{cases}u^{(N)}_{[N\xi]}(t), & N^{-1}\leqslant\xi<1\\ 0, & 0<\xi<N^{-1}\end{cases} \tag{7}$$

Then the following theorem is true.

Theorem 3. If conditions E and B hold then

$$\lim_{N\to+\infty}u^{(N)}(\xi,N^2t)=\frac{d\mu_t}{d\xi}(\xi),\qquad t>0$$

and the convergence is uniform in the variables (ξ, t, μ), where $\xi\in(0,1)$, $t\geqslant\epsilon>0$, $\mu\in M$. [$(d\mu_t/d\xi)(\xi)$ is the density of measure μ_t].

Consider the random walk on the lattice with nodes at $1/N$, $2/N,\ldots,(N-1)/N$ and with absorbing barriers at the points 0 and 1. Let the probability of transition for the time Δt from the node k/N to the node $(k-1)/N$ be $a_{k-1}\Delta t+o(\Delta t)$, from the node k/N to the node $(k+1)/N$ be $a_k\Delta t+o(\Delta t)$ and the probability to stay at the node k/N for this time interval be $1-(a_{k-1}+a_k)\Delta t+o(\Delta t)$. Let us assume that the trajectory of random process is right continuous. Then $A^{(N)}$ is the characteristic operator for this Markov process. Let $\|p^{(N)}_{k,m}(t)\|$ be the transition probability matrix for this Markov process.

Consider additionally the Brownian motion on the interval (0, 1) with absorbing barriers at the points 0 and 1 corresponding to characteristic Eq. (5) and let $p(\xi,\eta,t)$ be the transition function for this process (see Ref. 2). Define as above

$$p^{(N)}(\xi,\eta,t)=\begin{cases}Np^{(N)}_{[N\xi],[N\eta]}(t), & N^{-1}\leqslant\xi<1\ \text{ and }\ N^{-1}\leqslant\eta<1\\ 0, & 0<\xi<N^{-1}\ \text{ or }\ 0<\eta<N^{-1}\end{cases}$$

Theorem 4. If conditions E and B hold then at $t>0$

$$\lim_{N\to+\infty}p^{(N)}(\xi,\eta,N^2t)=p(\xi,\eta,t)$$

and the convergence is uniform in (ξ,η,t), where

$$(\xi,\eta)\in(0,1)\times(0,1),\qquad t\geqslant\epsilon>0$$

3. PROOFS OF THEOREMS 1 AND 2

Let us begin with the following lemma.

Lemma 1. Consider the chain of equations

$$a_{k-1}x_{k-1}-(a_{k-1}+a_k)x_k+a_kx_{k+1}=\lambda x_k \\ k=1,2,3,\ldots \tag{8}$$

where λ is a complex number. Let $x_k(\lambda)$, $k = 1, 2, 3, \ldots$, be the solution of this chain of equations with the initial conditions

$$x_0 = 0, \qquad x_1 = aa_0^{-1} \tag{9}$$

For $\xi \in [0, 1]$ assume that

$$F_N(\lambda, \xi) = N^{-1} x_{[N\xi]}(\lambda / N^2)$$

If condition E holds then

$$\lim_{N \to +\infty} F_N(\lambda, \xi) = (\lambda / a)^{-1/2} \sinh\left[\xi (\lambda / a)^{1/2}\right]$$

and the convergence is uniform in (λ, ξ), where λ belongs to a compact subset of the complex plane, $\xi \in [0, 1]$.

Proof. Let

$$x_k(\lambda) = \sum_{n=0}^{+\infty} x_{k,n} \lambda^n$$

where $x_{k,n}$ is independent of λ.

From (8) it follows that

$$a_l(x_{l+1,n} - x_{l,n}) = a_{l-1}(x_{l,n} - x_{l-1,n}) + (1 - \delta_n) x_{l,n-1}$$
$$l = 1, 2, 3, \ldots, \qquad n = 0, 1, 2, \ldots$$

where

$$\delta_n = \begin{cases} 1, & n = 0 \\ 0, & n > 0 \end{cases}$$

Summing these equations over $l = 1, 2, \ldots, m$ and using initial conditions (9), we have

$$a_m(x_{m+1,n} - x_{m,n}) = \delta_n a + (1 - \delta_n) \sum_{l=1}^{m} x_{l,n-1}$$

Divide these equations by a_m and sum them over $m = 1, 2, \ldots, k - 1$. Then using the initial conditions, we have

$$x_{k,n} = \delta_n a \sum_{m=0}^{k-1} a_m^{-1} + (1 - \delta_n) \sum_{m=1}^{k-1} a_m^{-1} \sum_{l=1}^{m} x_{l,n-1} \tag{10}$$
$$k = 2, 3, 4, \ldots$$

Now we prove that

$$\lim_{k \to +\infty} k^{-(2n+1)} x_{k,n} = \left[(2n + 1)! a^n\right]^{-1} \tag{11}$$

Condition E implies that

$$k^{-1} \sum_{m=0}^{k-1} a_m^{-1} = a^{-1} + \delta(k) \tag{12}$$

where $\delta(k)\to 0$ when $k\to +\infty$. Hence

$$\lim_{k\to+\infty} k^{-1}x_{k,0} = \lim_{k\to+\infty} k^{-1}a\sum_{m=0}^{k-1} a_m^{-1} = 1$$

and Eq. (11) holds if $n=0$.

We suppose that

$$k^{-(2n-1)}x_{k,n-1} = \left[(2n-1)!a^{n-1}\right]^{-1} + \epsilon(k)$$

where $\epsilon(k)\to 0$ when $k\to +\infty$. Using this inductive assumption and Eq. (12) we find from recurrent relation (10)

$$\begin{aligned}
k^{-(2n+1)}x_{k,n} &= k^{-(2n+1)}\sum_{l=1}^{k-1}\left(\sum_{m=0}^{k-1} a_m^{-1} - \sum_{m=0}^{l-1} a_m^{-1}\right)x_{l,n-1}\\
&= k^{-(2n+1)}\sum_{l=1}^{k-1}\left\{k\left(a^{-1}+\delta(k)\right) - l\left(a^{-1}+\delta(l)\right)\right\}l^{2n-1}\\
&\quad\times\left\{\left[(2n-1)!a^{n-1}\right]^{-1} + \epsilon(l)\right\}\\
&= k^{-(2n+1)}\sum_{l=1}^{k-1}(k-l)l^{2n-1}\left[(2n-1)!a^{n}\right]^{-1} + \theta(k)
\end{aligned}$$

$$\begin{aligned}
|\theta(k)| &= \Bigg|k^{-(2n+1)}\sum_{l=1}^{k-1}\Big\{(k-l)l^{2n-1}a^{-1}\epsilon(l) + \left[k\delta(k) - l\delta(l)\right]l^{2n-1}\\
&\quad\times\left(\left[(2n-1)!a^{n-1}\right]^{-1} + \epsilon(l)\right)\Big\}\Bigg|\\
&\leqslant a^{-1}k^{-1}\sum_{l=1}^{k-1}|\epsilon(l)| + \left[(2n-1)!a^{n-1}\right]^{-1}\\
&\quad\times\left\{|\delta(k)| + k^{-1}\sum_{l=1}^{k-1}|\delta(l)| + k^{-1}|\delta(k)|\sum_{l=1}^{k-1}|\epsilon(l)|\right.\\
&\quad\left. + k^{-1}\sum_{l=1}^{k-1}|\delta(l)\epsilon(l)|\right\}\to 0
\end{aligned}$$

when $k\to +\infty$. Hence

$$\begin{aligned}
\lim_{k\to+\infty} k^{-(2n+1)}x_{k,n} &= \left[(2n-1)!a^{n}\right]^{-1}\lim_{k\to+\infty} k^{-(2n+1)}\sum_{l=1}^{k-1}(k-l)l^{2n-1}\\
&= \left[(2n+1)!a^{n}\right]^{-1}
\end{aligned}$$

Thus, Eq. (11) holds for every n.

In a similar way by using condition E and recurrent relation (10) one can prove that there exist positive constants c_1 and c_2 such that

$$k^{-(2n+1)}x_{k,n} \leqslant c_1(n!c_2^n)^{-1}$$
$$k = 1,2,3,\dots, \qquad n = 0,1,2,\dots \tag{13}$$

Let $\xi \in [0,1]$. It follows from Eq. (11) that

$$\lim_{k\to+\infty} k^{-(2n+1)}x_{[k\xi],n} = \xi^{2n+1}((2n+1)!a^n)^{-1} \tag{14}$$

and the convergence is uniform in $\xi \in [\delta, 1]$, where δ is any positive number. Besides relation (13) implies that the functions $k^{-(2n+1)}x_{[k\xi],n}$ are equicontinuous at $\xi = 0$. Therefore in relation (14) the convergence is uniform in $\xi \in [0,1]$.

Consider now the series

$$F_k(\lambda,\xi) = \sum_{n=0}^{+\infty} k^{-(2n+1)}x_{[k\xi],n}\lambda^n$$

which contains only a finite set of nonvanishing terms. If $\xi \in [0,1]$ and $|\lambda| \leqslant \Lambda$ then it follows from relation (13) that this series is dominated by the convergent series

$$c_1 \sum_{n=0}^{+\infty} (n!c_2^n)^{-1}\Lambda^n$$

Hence

$$\lim_{k\to+\infty} F_k(\lambda,\xi) = \sum_{n=0}^{+\infty} \lim_{k\to+\infty} k^{-(2n+1)}x_{[k\xi],n}\lambda^n$$
$$= \sum_{n=0}^{+\infty} \xi^{2n+1}\left[(2n+1)!a^n\right]^{-1}\lambda^n = (\lambda/a)^{-1/2}\sinh\left[\xi(\lambda/a)^{1/2}\right]$$

and the convergence is uniform in variables (λ,ξ), where $|\lambda| \leqslant \Lambda$ and $\xi \in [0,1]$.

This concludes the proof of Lemma 1. ■

Proof of Theorem 1. Let $\Delta_k(\lambda)$ be the characteristic polynomial of operator $A^{(k)}$, $k = 2,3,4,\dots$. Then the sequence $\Delta_k(\lambda)$ satisfies the chain of recurrent equations

$$\Delta_{k+1} = -(a_{k-1} + a_k + \lambda)\Delta_k - a_{k-1}^2\Delta_{k-1}$$

with the initial conditions

$$\Delta_0 = 0, \qquad \Delta_1 = 1$$

We now set

$$x_k(\lambda) = a(a_0 a_1 \dots a_{k-1})^{-1}(-1)^{k-1}\Delta_k(\lambda), \qquad k = 2,3,4,\dots$$

Then $x_k(\lambda)$ is the solution to the chain of recurrent equations (8) with initial conditions (9). Therefore Lemma 1 implies

$$\lim_{k\to+\infty} a(a_0a_1\ldots a_{k-1})^{-1}(-1)^{k-1}k^{-1}\Delta_k(\lambda/k^2)$$
$$=\lim_{k\to+\infty} k^{-1}x_k(\lambda/k^2)=(\lambda/a)^{-1/2}\sinh\left[(\lambda/a)^{1/2}\right] \quad (15)$$

and the convergence is uniform in λ in compact subsets the complex plane.

Let $\lambda_1^{(k)},\lambda_2^{(k)},\ldots,\lambda_{k-1}^{(k)}$ be the eigenvalues of operator $A^{(k)}$ numbered in nonincreasing order. Then the roots of the polynomial $\Delta_k(\lambda/k^2)$ are $k^2\lambda_j^{(k)}$, $j=1,2,\ldots,k-1$. The limiting function $(\lambda/a)^{-1/2}\sinh[(\lambda/a)^{1/2}]$ vanishes at $\lambda_j=-a\pi^2j^2$, $j=1,2,3,\ldots$. Hence relation (15) and the argument principle (see, for example, Ref. 3) imply

$$\lim_{k\to+\infty} k^2\lambda_j^{(k)}=-a\pi^2j^2 \quad (16)$$

The argument principle also implies that for each j there exists a number $N(j)$ such that for all $k>N(j)$ the eigenvalue $\lambda_j^{(k)}$ is nondegenerated.

Let now $\varphi^{(N),j}$ be an eigenvector of the operator $A^{(N)}$ corresponding to the eigenvalue $\lambda_j^{(N)}$ such that $\varphi_1^{(N),j}\geqslant 0$ and

$$N^{-1}\sum_{k=1}^{N-1}\left(\varphi_k^{(N),j}\right)^2=1$$

Equations (8) and (9) imply that the vector

$$\left(x_1(\lambda_j^{(N)}),x_2(\lambda_j^{(N)}),\ldots,x_{N-1}(\lambda_j^{(N)})\right)\in\mathbb{R}^{N-1}$$

is an eigenvector of the operator $A^{(N)}$ corresponding to the eigenvalue $\lambda_j^{(N)}$. We shall suppose that N is large enough and therefore $\lambda_j^{(N)}$ is a nondegenerated eigenvalue. Then we have

$$\varphi_k^{(N),j}=N^{-1}C_j^{(N)}x_k\left(\lambda_j^{(N)}\right),\qquad k=1,2,\ldots,N-1$$

where

$$\left(C_j^{(N)}\right)^2=N^{-1}\sum_{k=1}^{N-1}\left[N^{-1}x_k\left(\lambda_j^{(N)}\right)\right]^2=\int_0^1\left[F_N\left(N^2\lambda_j^{(N)},\xi\right)\right]^2d\xi$$

and according to definition (3)

$$\varphi^{(N),j}(\xi)=\left\{\int_0^1\left[F_N\left(N^2\lambda_j^{(N)},\xi\right)\right]^2d\xi\right\}^{-1/2}F_N\left(N^2\lambda_j^{(N)},\xi\right)$$

Let

$$F(\lambda,\xi)=(\lambda/a)^{-1/2}\sinh\left[\xi(\lambda/a)^{1/2}\right]$$

Then using Lemma 1 and Eq. (16) we obtain

$$\begin{aligned}\lim_{N\to+\infty} F_N\left(N^2\lambda_j^{(N)},\xi\right) &= F(\lambda_j,\xi)\\ &\quad+\lim_{N\to+\infty}\left\{F_N\left(N^2\lambda_j^{(N)},\xi\right)-F\left(N^2\lambda_j^{(N)},\xi\right)\right\}\\ &\quad+\lim_{N\to+\infty}\left\{F\left(N^2\lambda_j^{(N)},\xi\right)-F(\lambda_j,\xi)\right\}\\ &= F(\lambda_j,\xi)\end{aligned}$$

uniformly in $\xi\in(0,1)$, where $\lambda_j = -a\pi^2 j^2$. Hence

$$\lim_{N\to+\infty}\varphi^{(N),j}(\xi)=\left\{\int_0^1\left[F(\lambda_j,\xi)\right]^2 d\xi\right\}^{-1/2}F(\lambda_j,\xi)=\sqrt{2}\sin(\pi j\xi)$$

uniformly in $\xi\in(0,1)$.

This concludes the proof of Theorem 1. ■

Proof of Theorem 2. The numbers $r_{k,m}^{(N)}$, $1\leqslant k\leqslant N-1$, $1\leqslant m\leqslant N-1$ satisfy the system of equations

$$a_{k-1}r_{k-1,m}^{(N)}-(a_{k-1}+a_k)r_{k,m}^{(N)}+a_k r_{k+1,m}^{(N)}=\delta_{k,m}$$
$$r_{0,m}^{(N)}=r_{N,m}^{(N)}=0$$

where $k=1,2,\ldots,N-1$, $m=1,2,\ldots,N-1$,

$$\delta_{k,m}=\begin{cases}1, & k=m\\ 0, & k\neq m\end{cases}$$

Solving this system of equations in a similar way as in Lemma 1 we obtain

$$r_{k,m}^{(N)}=a_0 r_{1,m}^{(N)}\sum_{l=0}^{k-1}a_l^{-1}+\theta(k-m)\sum_{l=m}^{k-1}a_l^{-1}$$
$$r_{0,m}^{(N)}=r_{N,m}^{(N)}=0$$

where

$$\theta(k)=\begin{cases}1, & k>0\\ 0, & k\leqslant 0\end{cases}$$

The condition $r_{N,m}^{(N)}=0$ implies that

$$r_{1,m}^{(N)}=-\left(a_0\sum_{l=0}^{N-1}a_l^{-1}\right)^{-1}\sum_{l=m}^{N-1}a_l^{-1}$$

and, finally,

$$r_{k,m}^{(N)}=\theta(k-m)\sum_{l=m}^{k-1}a_l^{-1}-\left(\sum_{l=0}^{N-1}a_l^{-1}\right)^{-1}\sum_{l=m}^{N-1}a_l^{-1}\sum_{l=0}^{k-1}a_l^{-1}$$

Therefore

$$\lim_{N\to+\infty} N^{-1}r^{(N)}_{[N\xi],[N\eta]} = \theta(\xi-\eta)(\xi-\eta)a^{-1} + (1-\eta)\xi a^{-1}$$
$$= a^{-1}\{\xi\eta - \min(\xi,\eta)\}$$

and the convergence is uniform in $(\xi,\eta)\in(0,1)\times(0,1)$.

4. THE PROOFS OF THEOREMS 3 AND 4

First we need uniform estimations for the eigenvalues and the eigenvectors of the operators $A^{(N)}$.

Lemma 2. If condition B holds then there exists a constant $c_3 > 0$ such that $N^2\lambda_j^{(N)} \leqslant -c_3 j^2$.

Proof. Let $\tilde{A}^{(N)}$ be the operator defined as $A^{(N)}$ in the case when all $a_k = c$. Then $A^{(N)} \leqslant \tilde{A}^{(N)}$.

Indeed, for any $u^{(N)}\in\mathbb{R}^{N-1}$ we have

$$\sum_{k=1}^{N-1}(A^{(N)}u^{(N)})_k u_k^{(N)} = -a_0\big(u_1^{(N)}\big)^2 - \sum_{k=1}^{N-2} a_k\big(u_{k+1}^{(N)} - u_k^{(N)}\big)^2 - a_{N-1}\big(u_{N-1}^{(N)}\big)^2$$
$$\leqslant -c\big(u_1^{(N)}\big)^2 - \sum_{k=1}^{N-2} c\big(u_{k+1}^{(N)} - u_k^{(N)}\big)^2 - c\big(u_{N-1}^{(N)}\big)^2$$
$$= \sum_{k=1}^{N-1}(\tilde{A}^{(N)}u^{(N)})_k u_k^{(N)}.$$

Since the eigenvalues of operator $\tilde{A}^{(N)}$ equal $-4c\sin^2(\pi j/2N)$, $j = 1, 2, \ldots, N-1$, then $\lambda_j^{(N)} \leqslant -4c\sin^2(\pi j/2N)$. From the inequality $\sin x \geqslant 2x/\pi$ which is true for $x\in[0,\pi/2]$ it follows that

$$\lambda_j^{(N)} \leqslant -4c\sin^2(\pi j/2N) \leqslant -4cj^2/N^2$$

This concludes the proof. ∎

Lemma 3. Let $\varphi^{(N),j}$ be an eigenvector of operator $A^{(N)}$ corresponding to the eigenvalue $\lambda_j^{(N)}$. If condition E holds then there exists the constant c_4 such that

$$\max_{k=1,2,\ldots,N-1}\left|\varphi_k^{(N),j}\right| \leqslant -c_4N^2\lambda_j^{(N)}\left(N^{-1}\sum_{k=1}^{N-1}\left(\varphi_k^{(N),j}\right)^2\right)^{1/2}$$

Proof. Using definition (3) one can rewrite the last inequality in the following way:

$$\sup_{\xi\in(0,1)}\left|\varphi^{(N),j}(\xi)\right| \leqslant -c_4N^2\lambda_j^{(N)}\left\{\int_0^1\left[\varphi^{(N),j}(\xi)\right]^2 d\xi\right\}^{1/2}$$

The vector $\varphi^{(N),j}$ is the solution of the equation

$$\varphi^{(N),j} = \lambda_j (A^{(N)})^{-1} \varphi^{(N),j}$$

Therefore, according to definition (4) we have

$$\varphi^{(N),j}(\xi) = N^2 \lambda_j^{(N)} \int_0^1 r^{(N)}(\xi,\eta) \varphi^{(N),j}(\eta)\, d\eta \tag{17}$$

As follows from Theorem 2, there exists the constant c_4 such that $|r^{(N)}(\xi,\eta)| \leqslant c_4$. Applying this estimate to Eq. (17) we obtain

$$\begin{aligned}\left|\varphi^{(N),j}(\xi)\right| &\leqslant -c_4 N^2 \lambda_j^{(N)} \int_0^1 \left|\varphi^{(N),j}(\eta)\right| d\eta \\ &\leqslant -c_4 N^2 \lambda_j^{(N)} \left\{ \int_0^1 \left[\varphi^{(N),j}(\eta)\right]^2 d\eta \right\}^{1/2}\end{aligned}$$

This concludes the proof. ■

Proof of Theorem 3. Let

$$(\mu^{(N)}, \varphi^{(N),j}) = \int_0^1 \varphi^{(N),j}(\xi)\, d\mu(\xi), \qquad (\mu, \varphi^j) = \sqrt{2} \int_0^1 \sin(\pi j \xi)\, d\mu(\xi)$$

According to definitions (3) and (6)

$$\int_0^1 \varphi^{(N),j}(\xi)\, d\mu(\xi) = N^{-1} \sum_{k=1}^{N-1} \mu_k^{(N)} \varphi_k^{(N),j}$$

Using notations (7) we obtain

$$u^{(N)}(\xi, N^2 t) = \sum_{j=1}^{N-1} (\mu^{(N)}, \varphi^{(N),j}) \varphi^{(N),j}(\xi) e^{N^2 \lambda_j^{(N)} t} \tag{18}$$

Besides (see, for example, Ref. 4)

$$\frac{d\mu_t}{d\xi}(\xi) = \sum_{j=1}^{+\infty} (\mu, \varphi^j) \varphi^j(\xi) e^{\lambda_j t}$$

where $\varphi^j(\xi) = \sqrt{2}\sin(\pi j \xi)$, $\lambda_j = -a\pi^2 j^2$.

It follows from Lemmas 2 and 3 that

$$\begin{aligned}\left|(\mu^{(N)}, \varphi^{(N),j}) \varphi^{(N),j}(\xi) e^{N^2 \lambda_j^{(N)} t}\right| &\leqslant c_4^2 N^4 \left(\lambda_j^{(N)}\right)^2 e^{N^2 \lambda_j^{(N)} t} \\ &\leqslant c_4^2 t^{-2} \left(N^2 \lambda_j^{(N)} t\right)^2 e^{N^2 \lambda_j^{(N)} t} \\ &\leqslant c_4 \epsilon^{-2} \left(c_3 j^2 \epsilon\right)^2 e^{-c_3 j^2 \epsilon} \\ &= c_3^2 c_4^2 j^2 e^{-c_3 j^2 \epsilon}\end{aligned}$$

if $t \geqslant \epsilon > 0$ and $j^2 \geqslant 2/c_3\epsilon$. Therefore, if $t \geqslant \epsilon$ the series (18) is dominated by the convergent series

$$c_3^2 c_4^2 \sum_{j=1}^{+\infty} j^2 e^{-c_3 j^2 \epsilon}$$

Using Theorem 1 we obtain for $t > 0$

$$\lim_{N\to+\infty} u^{(N)}(\xi, N^2 t) = \sum_{j=1}^{+\infty} \lim_{N\to+\infty} \left\{ (\mu^{(N)}, \varphi^{(N),j}) \varphi^{(N),j}(\xi) e^{N^2 \lambda_j^{(N)} t} \right\}$$
$$= \sum_{j=1}^{+\infty} (\mu, \varphi^j) \varphi^j(\xi) e^{\lambda_j t} = \frac{d\mu_t}{d\xi}(\xi)$$

and the convergence is uniform in (ξ, t, μ), where $\xi \in (0, 1)$, $t \geqslant \epsilon > 0$, $\mu \in M$.

This concludes the proof of Theorem 3. ■

Proof of Theorem 4. Let $\mu(\xi)$ be the δ measure at the point $\xi \in (0, 1)$. Then if $t > 0$

$$\frac{d\mu(\xi)_t}{d\eta}(\eta) = p(\xi, \eta, t)$$

Besides the vector $(p_{k,1}^{(N)}(t), p_{k,2}^{(N)}(t), \ldots, p_{k,N-1}^{(N)}(t)) \in \mathbb{R}^{N-1}$ is the solution of Eq. (1) with the initial condition $p_{k,m}^{(N)}(0) = \delta_{k,m}$. Therefore this theorem directly follows from Theorem 3. ■

Note. We would like to emphasize the decisive role of a computer experiment in the formulation of Theorem 1.

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